Cartesian Products of Regular Graphs are Antimagic

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Abstract

An *antimagic labeling* of a finite undirected simple graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labeling. In 1990, Hartsfield and Ringel [4] conjectured that every simple connected graph, but K_2, is antimagic. In this article, we prove that a new class of Cartesian product graphs are antimagic. In addition, by combining this result and the antimagicness result on toroidal grids (Cartesian products of two cycles) in [6], all Cartesian products of two or more regular graphs can be proved to be antimagic.

Keywords: *antimagic; magic; labeling; regular graph; Cartesian product*

1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the notation and terminology of [4]. In 1990, Hartsfield and Ringel [4] introduced the concept of *antimagic* graph. An *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling. Hartsfield and Ringel showed that paths $P_n (n \geq 3)$, cycles, wheels, and complete graphs $K_n (n \geq 3)$ are antimagic. They conjectured that all trees except K_2 are antimagic. Moreover, all connected graphs except K_2 are antimagic. These two conjectures are unsettled. In [2], Alon et al showed that the latter conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$. They also proved that complete partite graphs (other than K_2) and n-vertex graphs with maximum degree at least $n - 2$ are antimagic. In [5], Hefetz proved several special cases and variants of the latter conjecture, the main tool used is the Combinatorial NullStellenSatz (see [1]). In [6], Wang showed that the toroidal grids, i.e., Cartesian products of two or more cycles, are antimagic.

In this paper, we prove that the Cartesian products $G_1 \times G_2$ of a regular graph G_1 and a graph G_2 of bounded degrees are antimagic, provided that the degrees of G_1 and G_2 satisfy some inequality. By combining this result and the antimagicness result on the Cartesian products of two cycles in [6], all Cartesian products of two or more regular graphs (not necessarily connected) can be proved to be antimagic. First, we introduce another concept about graph labeling called *δ-approximately magic*.

Definition 1.1 A *δ-approximately magic labeling* of a graph with m edges is a bijection from the set of edges to the integers $1, \ldots, m$ such that the difference between the largest and the smallest vertex sums is at most δ, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called *δ-approximately magic* if it has a δ-approximately magic labeling.

Thus 0-approximately magic is the same as *magic* in [4], or *supermagic* in some literature. We first prove some approximately magicness results on connected regular graphs, the following is proved in Section 2.

Theorem 1.1 *If G is an n-vertex k-regular connected graph ($k \geq 1$), then G is $(\frac{nk}{2}-1)$-approximately magic in case k is odd, k-approximately magic in case k is even.*

Recall that the *Cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph with vertex set $V_1 \times V_2$, and (u_1, u_2) is adjacent to (v_1, v_2) in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2 v_2 \in E_2$, or, $u_2 = v_2$ and $u_1 v_1 \in E_1$.

Using the approximately magicness results in Theorem 1.1, we prove the following theorem in Section 3.

Theorem 1.2 *If G_1 is an n_1-vertex k_1-regular connected graph, and G_2 is a graph (not necessarily connected) with maximum degree at most k_2, minimum degree at least one, then $G_1 \times G_2$ is antimagic, provided that k_1 is odd and $\frac{k_1^2-k_1}{2} \geq k_2$, or, k_1 is even and $\frac{k_1^2}{2} \geq k_2$ and k_1, k_2 are not both equal to 2.*

By combining Theorem 1.2 and the antimagicness result on the Cartesian products of two cycles in [6], the following theorem is obtained in Section 4.

Theorem 1.3 *All Cartesian products of two or more regular graphs (not necessarily connected) are antimagic.*

Finally, we give a generalization of Theorems 1.1 in which G is not necessarily connected, and a generalization of Theorem 1.2 in which G_1 is not necessarily connected. The following two theorems are proved in Section 5.

Theorem 1.4 (generalization of Theorem 1.1) *If G is an n-vertex k-regular graph ($k \geq 1$, G is not necessarily connected), then G is $(\frac{nk}{2} - 1)$-approximately magic in case k is odd, $(\frac{2n}{3} + k - 1)$-approximately magic in case k is even.*

Theorem 1.5 (generalization of Theorem 1.2) *If G_1 is an n_1-vertex k_1-regular graph, and G_2 is a graph with maximum degree at most k_2, minimum degree at least one (G_1,G_2 are not necessarily connected), then $G_1 \times G_2$ is antimagic, provided that k_1 is odd and $\frac{k_1^2-k_1}{2} \geq k_2$, or, k_1 is even and $\frac{k_1^2}{2} > k_2$.*

For more results, open problems and conjectures on magic graphs, antimagic graphs and various graph labeling problems, please see [3].

Throughout the paper, we denote by $\lceil x \rceil$ (ceiling of x) the least integer that is not less than x, denote by $\lfloor x \rfloor$ (floor of x) the largest integer that is not greater than x.

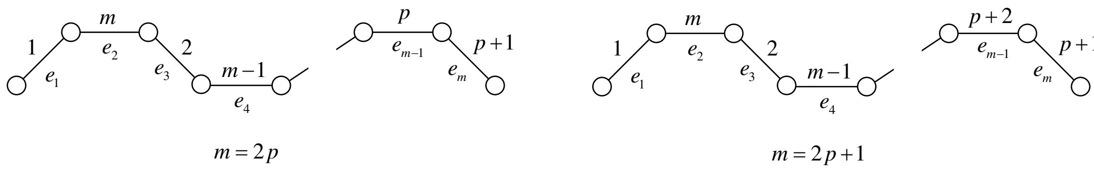

Fig. 1. Labeling of the sequence of trails $T : t_1 t_2 \dots t_{\frac{n}{2}}$.

2 Proof of Theorem 1.1

We begin with some terms and definitions (see [4]). A *walk* in a graph G is an alternating sequence $v_1 e_1 v_2 e_2 \cdots e_{t-1} v_t$ of vertices and edges of G, with the property that every edge e_i is incident with v_i and v_{i+1}, for $i = 1, \dots, t-1$. Vertices and edges may be repeated in a walk. A *trail* in a graph G is a walk in G with the property that no edge is repeated. A *circuit* is a closed trail, that is a trail whose endpoints are the same vertex. A *cycle* is a circuit with the property that no vertex is repeated. An *Eulerian circuit* in a graph G is a circuit that contains every edge of G. In order to prove Theorem 1.1 for the case that k is odd, we need the following theorem ([4], pp. 56),

Theorem 2.1 (part of Listing Theorem). *If G is a connected graph with precisely $2h$ vertices of odd degree, $h \neq 0$, then there exist h trails in G such that each edge of G is in exactly one of these trails.*

If G is a connected n-vertex regular graph of odd degree k, by Theorem 2.1, there are $n/2$ trails $t_1, t_2, \dots, t_{\frac{n}{2}}$ in G, such that each edge of G is in exactly one of these trails. Denote $|t|$ to be the length (number of edges) of a trail t. Without loss of generality, assume that $|t_1| \geq |t_2| \geq \dots \geq |t_{\frac{n}{2}}|$. By concatenating these trails we get a sequence $T : t_1 t_2 \dots t_{\frac{n}{2}}$, which contains all the m $(= \frac{nk}{2})$ edges of G. Number the edges of G according to their ordering in T, let $e_1, e_2, \dots, e_m$ be the numbering. Assign the labels $1, 2, \dots, \lceil \frac{m}{2} \rceil$ to the edges of odd indices $e_1, e_3, \dots$ etc., and assign the labels $m, m-1, \dots, \lceil \frac{m}{2} \rceil + 1$ to the edges of even indices $e_2, e_4, \dots$ etc. (see Figure 1). It is easy to see that for the above labeling, the sum of any two consecutive edges in T is either $m+1$ or $m+2$. In addition, if e is the first or the last edge of a trail, then the largest possible label received by e is at most $m - \frac{k-1}{2}$ (notice that $|t_1| \geq k$). For each vertex v of G, the k edges incident with v can be partitioned into $\frac{k-1}{2}$ pairs and a singleton, such that each pair is composed of two consecutive edges within one of the above $n/2$ trails, and the single edge is the first or the last edge of a trail. Therefore, for the above labeling, the sum received by any vertex of G is at most $(m - \frac{k-1}{2}) + \frac{k-1}{2} \times (m+2) = m + \frac{k-1}{2} \times (m+1)$, at least $1 + \frac{k-1}{2} \times (m+1)$, implying that this is an $(\frac{nk}{2} - 1)$-approximately magic labeling of G. For the case that the degree k is even, we need the following lemma.

Lemma 2.2 *Every m-vertex connected regular graph of degree 2 (i.e., cycle C_m) is 2-approximately magic, for $m \geq 3$.*

Proof: We have the following four cases:

Case 1. $m \equiv 1 \pmod 4$. Let $m = 4t+1$, $t \geq 1$. Partition the labels $1, 2, \dots, m$ into $2t+1$ groups $(1), (2,3), \dots, (2t, 2t+1), (2t+2, 2t+3), \dots, (m-1, m)$. First assign label 1 to an arbitrary edge of C_m, then assign the labels $(m, m-1), (2,3), (m-2, m-3), (4,5), \dots, (2t, 2t+1)$ in a way that

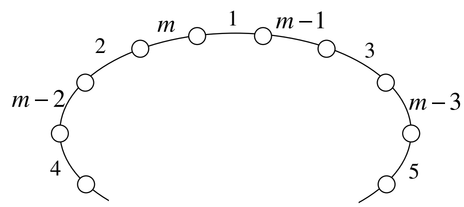

Fig. 2. 2-Approximately magic labeling of C_m

each pair of labels are assigned to the two edges that have common endpoints with the labeled arc.

Case 2. $m \equiv 3 \pmod 4$. Let $m = 4t+3$, $t \geq 0$. Partition the labels $1, 2, \ldots, m$ into $2t+2$ groups $(1), (2,3), \ldots, (2t, 2t+1), (2t+2, 2t+3), \ldots, (m-1, m)$. First assign label 1 to an arbitrary edge of C_m, then assign the labels $(m, m-1), (2,3), (m-2, m-3), (4,5), \ldots, (2t+3, 2t+2)$ in the same way as in Case 1.

Case 3. $m \equiv 0 \pmod 4$. Let $m = 4t+4$, $t \geq 0$. Partition the labels $1, 2, \ldots, m$ into $2t+3$ groups $(1), (2,3), \ldots, (2t, 2t+1), (2t+2), (2t+3, 2t+4), \ldots, (m-1, m)$. First assign label 1 to an arbitrary edge of C_m, then assign the labels $(m, m-1), (2,3), (m-2, m-3), (4,5), \ldots, (2t+4, 2t+3)$ in the way that each pair of labels are assigned to the two edges that have common endpoints with the labeled arc, finally assign the label $2t+2$ to the one non-labeled edge.

Case 4. $m \equiv 2 \pmod 4$. Let $m = 4t+2$, $t \geq 1$. Partition the labels $1, 2, \ldots, m$ into $2t+2$ groups $(1), (2,3), \ldots, (2t, 2t+1), (2t+2), (2t+3, 2t+4), \ldots, (m-1, m)$. First assign label 1 to an arbitrary edge of C_m, then assign the labels $(m, m-1), (2,3), (m-2, m-3), (4,5), \ldots, (2t, 2t+1), (2t+2)$ in the same way as in Case 3.

It is easy to see that in any of the above cases, the vertex sums of C_m are all among $m, m+1$, and $m+2$, implying the assertion of the lemma (see Figure 2). ■

Recall that a connected graph with all vertices of even degrees has an Eulerian circuit. It follows that if G is a connected n-vertex regular graph of even degree k, G has an Eulerian circuit, without loss of generality, say $e_1 e_2 \ldots e_m$, where $m = \frac{nk}{2}$. We label $1, 2, \ldots, m$ to this circuit using the above 2-approximately magic labeling in Lemma 2.2 (here we view this circuit as a cycle). For each vertex v of G, the k edges incident with v can be partitioned into $k/2$ pairs such that each pair is composed of two consecutive edges in the Eulerian circuit $e_1 e_2 \ldots e_m$, thus the sum of each pair is among $m, m+1$, and $m+2$. Therefore, for the above labeling, the sum received by any vertex of G is at least $\frac{k}{2} \times m$, at most $\frac{k}{2} \times (m+2)$, implying that this is a k-approximately magic labeling of G.

3 Proof of Theorem 1.2

Suppose that G_1 is an n_1-vertex k_1-regular connected graph, $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, and G_2 is a graph with maximum degree at most k_2, minimum degree at least one, $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Denote by m_1 $(= \frac{k_1 n_1}{2})$ and m_2 the number of edges of G_1 and G_2, respectively.

Let $f : E(G_1 \times G_2) \to \{1, 2, \ldots, m_2 n_1 + m_1 n_2\}$ be an edge labeling of $G_1 \times G_2$, and denote the induced sum at vertex (u, v) by $w(u, v) = \sum f((u, v), (y, z))$, where the sum runs over all vertices

(y,z) adjacent to (u,v) in $G_1 \times G_2$. In the product graph $G_1 \times G_2$, at each vertex (u,v), the edges incident to this vertex can be partitioned into two parts, one part is contained in a copy of G_1 component, and the other part is contained in a copy of G_2 component. Denote by $w_1(u,v)$ and $w_2(u,v)$ the sum at vertex (u,v) restricted to G_1 component and G_2 component respectively, i.e., $w_1(u,v) = \sum f((u,v),(y,v))$, where the sum runs over all vertices y adjacent to u in G_1, and $w_2(u,v) = \sum f((u,v),(u,z))$, where the sum runs over all vertices z adjacent to v in G_2. Therefore, $w(u,v) = w_1(u,v) + w_2(u,v)$.

Given two isomorphic graphs G and G', and two labelings f and f' of G and G' respectively, we call f' is a *δ-shift* of f, if for each edge $e \in E(G)$ and its counterpart $e' \in E(G')$ under the isomorphism, we have $f'(e') = f(e) + \delta$. Now we will present our labeling of $G_1 \times G_2$, which contains two steps.

Step 1 (renaming vertices): Assign labels $1, 2, \ldots, m_1$ to the edges of G_1, such that the labeling is $(\frac{n_1 k_1}{2} - 1)$-approximately magic if k_1 is odd, k_1-approximately magic if k_1 is even. Without loss of generality, we can rename the vertices of G_1 such that $w(u_1) \leq w(u_2) \leq \ldots \leq w(u_{n_1})$, denote this labeling by L_1. Assign labels $1, n_1+1, 2n_1+1, \ldots, (m_2-1)n_1+1$ arbitrarily to the edges of G_2. Similarly, rename the vertices of G_2 such that $w(v_1) \leq w(v_2) \leq \ldots \leq w(v_{n_2})$, denote this labeling by L_2.

Step 2 (labeling on $G_1 \times G_2$): Assign labels $m_2n_1+1, m_2n_1+2, \ldots, m_2n_1+m_1n_2$ to the edges that are contained in copies of G_1 component. For the i-th G_1 component (with vertices (u_1,v_i), (u_2,v_i),$\ldots$, (u_{n_1},v_i)), label its edges with $m_2n_1+(i-1)m_1+1, m_2n_1+(i-1)m_1+2, \ldots, m_2n_1+(i-1)m_1+m_1$, such that the labeling is an $[m_2n_1+(i-1)m_1]$-shift of L_1, under the natural isomorphism, for $i = 1, \ldots, n_2$. Since G_1 is regular, we have $w_1(u_1,v_i) \leq w_1(u_2,v_i) \leq \ldots \leq w_1(u_{n_1},v_i)$, for $i = 1, \ldots, n_2$.

Assign labels $1, 2, \ldots, m_2n_1$ to the edges that are contained in copies of G_2 component. For the j-th G_2 component (with vertices (u_j,v_1), (u_j,v_2),$\ldots$, (u_j,v_{n_2})), label its edges with $j, n_1+j, 2n_1+j, \ldots, (m_2-1)n_1+j$, such that the labeling is a $(j-1)$-shift of L_2, under the natural isomorphism, for $j = 1, \ldots, n_1$. From the way we name the vertices of G_2, we have $w_2(u_1,v_1) \leq w_2(u_1,v_2) \leq \ldots \leq w_2(u_1,v_{n_2})$.

In what follows we will prove that for the above labeling, if k_1 is odd and $\frac{k_1^2-k_1}{2} \geq k_2$, or, if k_1 is even and $\frac{k_1^2}{2} \geq k_2$ and k_1, k_2 are not both equal to 2, then

$$\begin{aligned} & w(u_1,v_1) < w(u_2,v_1) < \ldots\ldots\ldots\ldots < w(u_{n_1},v_1) < \\ & w(u_1,v_2) < w(u_2,v_2) < \ldots\ldots\ldots\ldots < w(u_{n_1},v_2) < \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & w(u_1,v_{n_2}) < w(u_2,v_{n_2}) < \ldots\ldots\ldots\ldots < w(u_{n_1},v_{n_2}), \end{aligned} \tag{1}$$

implying that the above labeling is antimagic.

For each $i \in \{1, \ldots, n_2\}$, we have $w_1(u_1,v_i) \leq w_1(u_2,v_i) \leq \ldots \leq w_1(u_{n_1},v_i)$, and $w_2(u_1,v_i) < w_2(u_2,v_i) < \ldots < w_2(u_{n_1},v_i)$ since $w_2(u_{j+1},v_i) - w_2(u_j,v_i) = d(v_i)$, where $d(v_i) \geq 1$ is the degree of v_i in G_2, $j = 1, \ldots, n_1-1$. It follows that $w(u_1,v_i) < w(u_2,v_i) < \ldots < w(u_{n_1},v_i)$, for $i = 1, \ldots, n_2$. In order to prove $w(u_1,v_{i+1}) > w(u_{n_1},v_i)$, for $i = 1, \ldots, n_2-1$, there are two cases.

Case 1. k_1 is odd. For each $i \in \{1, \ldots, n_2-1\}$, we have $w(u_1,v_{i+1}) \geq w(u_1,v_i) + \frac{n_1k_1^2}{2}$ since $w_1(u_1,v_{i+1}) = w_1(u_1,v_i) + m_1k_1 = w_1(u_1,v_i) + \frac{n_1k_1^2}{2}$ (notice that the labeling on the $(i+1)$-th G_1

component is an m_1-shift of the labeling on the i-th G_1 component) and $w_2(u_1, v_{i+1}) \geq w_2(u_1, v_i)$. In addition, we have $w(u_{n_1}, v_i) \leq w(u_1, v_i) + (\frac{n_1k_1}{2} - 1) + k_2(n_1 - 1)$ since $w_1(u_{n_1}, v_i) \leq w_1(u_1, v_i) + (\frac{n_1k_1}{2} - 1)$ (notice that G_1 is regular and L_1 is $(\frac{n_1k_1}{2} - 1)$-approximately magic when k_1 is odd), and $w_2(u_{n_1}, v_i) = w_2(u_1, v_i) + d(v_i)(n_1 - 1) \leq w_2(u_1, v_i) + k_2(n_1 - 1)$. It follows that $w(u_1, v_{i+1}) - w(u_{n_1}, v_i) \geq (w(u_1, v_i) + \frac{n_1k_1^2}{2}) - (w(u_1, v_i) + (\frac{n_1k_1}{2} - 1) + k_2(n_1 - 1)) = n_1(\frac{k_1^2 - k_1}{2} - k_2) + 1 + k_2 > 0$, for $i = 1, \ldots, n_2 - 1$.

Case 2. k_1 is even. Similarly, for each $i \in \{1, \ldots, n_2 - 1\}$, we have $w(u_1, v_{i+1}) \geq w(u_1, v_i) + \frac{n_1k_1^2}{2}$ since $w_1(u_1, v_{i+1}) = w_1(u_1, v_i) + m_1k_1 = w_1(u_1, v_i) + \frac{n_1k_1^2}{2}$ and $w_2(u_1, v_{i+1}) \geq w_2(u_1, v_i)$. In addition, $w(u_{n_1}, v_i) \leq w(u_1, v_i) + k_1 + k_2(n_1 - 1)$ holds since $w_1(u_{n_1}, v_i) \leq w_1(u_1, v_i) + k_1$ (L_1 is k_1-approximately magic when k_1 is even) and $w_2(u_{n_1}, v_i) = w_2(u_1, v_i) + d(v_i)(n_1 - 1) \leq w_2(u_1, v_i) + k_2(n_1 - 1)$. It follows that $w(u_1, v_{i+1}) - w(u_{n_1}, v_i) \geq (w(u_1, v_i) + \frac{n_1k_1^2}{2}) - (w(u_1, v_i) + k_1 + k_2(n_1 - 1)) = n_1(\frac{k_1^2}{2} - k_2) + k_2 - k_1$.

If $\frac{k_1^2}{2} > k_2$, since k_1 is even, $\frac{k_1^2}{2} - k_2 \geq 1$, then $w(u_1, v_{i+1}) - w(u_{n_1}, v_i) \geq n_1(\frac{k_1^2}{2} - k_2) + k_2 - k_1 \geq n_1 + k_2 - k_1 > 0$ (since $n_1 > k_1$). If $\frac{k_1^2}{2} = k_2$, since k_1, k_2 are not both equal to 2, we have $k_1 > 2$, it follows that $w(u_1, v_{i+1}) - w(u_{n_1}, v_i) \geq k_2 - k_1 = \frac{k_1^2}{2} - k_1 > 0$. Thus, in any case, we have $w(u_1, v_{i+1}) - w(u_{n_1}, v_i) > 0$, for $i = 1, \ldots, n_2 - 1$.

Therefore, (1) holds, implying the assertion of Theorem 1.2.

4 Proof of Theorem 1.3

Since the Cartesian product preserves regularity, we only need to prove that all Cartesian products of two regular graphs are antimagic. We first prove Theorem 1.3 for the case that G_1 and G_2 are both connected, then we generalize the proof to the case where G_1 and G_2 are not necessarily connected.

4.1 Connected Case

Suppose that G_1 is an n_1-vertex k_1-regular connected graph, and G_2 is an n_2-vertex k_2-regular connected graph. Without loss of generality, assume that $k_1 \geq k_2$. Furthermore, we may assume $k_1 \geq 2$ since $K_2 \times K_2$ can be easily verified as antimagic. If $k_1 = 2$ and $k_2 = 1$, by Theorem 1.2, $G_1 \times G_2$ is antimagic. If $k_1 = 2$ and $k_2 = 2$, then $G_1 \times G_2$ is a toroidal grid graph and its antimagicness is proved in [6]. For $k_1 \geq 3$, if k_1 is odd, then $\frac{k_1^2 - k_1}{2} \geq k_1 \geq k_2$; if k_1 is even, then $k_1 \geq 4$, $\frac{k_1^2}{2} > k_1 \geq k_2$. Thus by Theorem 1.2, $G_1 \times G_2$ is antimagic.

4.2 Unconnected Case

Denote by c_1 and c_2 the numbers of connected components of G_1 and G_2, respectively. It is easy to see that the number of connected components of $G_1 \times G_2$ is $c = c_1 \times c_2$, and each of its connected components is a $(k_1 + k_2)$-regular graph (which is product of one k_1-regular connected graph and one k_2-regular connected graph). Let $m_1, m_2, \ldots, m_c$ be the numbers of edges of these connected components $C_1, C_2, \ldots, C_c$. The labeling of $G_1 \times G_2$ goes as follows. Assign $1, 2, \ldots, m_1$

to the edges of C_1, assign $m_1+1, m_1+2, \ldots, m_1+m_2$ to the edges of C_2, ..., and assign $m_1+\ldots+m_{c-1}+1, m_1+\ldots+m_{c-1}+2, \ldots, m_1+\ldots+m_{c-1}+m_c$ to the edges of C_c, such that the labeling of each connected component is antimagic (this can be achieved because of the previous proof for the case where G_1 and G_2 are both connected and the regularity of each component). The whole labeling of $G_1 \times G_2$ is antimagic, since between any two different components, any sum of k_1+k_2 labels from a group of larger labels must be greater than any sum of k_1+k_2 labels from a group of smaller labels.

5 Generalizations of Theorem 1.1 and 1.2

In this section, we will prove Theorems 1.4, a generalization of Theorem 1.1 in which G is not necessarily connected, and Theorem 1.5, a generalization of Theorem 1.2 in which G_1 is not necessarily connected.

5.1 Proof of Theorem 1.4

For the case k is odd, by Theorem 2.1 (Listing), for each connected component of G (which is a connected k-regular graph), if it has n_i vertices, we can decompose it into $\frac{n_i}{2}$ trails. By running this decomposition over all connected components of G, we can get a total number of $\frac{n}{2}$ trails, such that each edge of G is in exactly one of these trails. It is easy to see that the largest length of these trails is at least k. We concatenate these trails into a sequence in the ordering of nonincreasing lengths, and label the sequence in the same way as in Theorem 1.1, which results in an $(\frac{nk}{2}-1)$-approximately magic labeling of G. For the case k is even, we first prove the following lemma.

Lemma 5.1 *If G is an n-vertex graph consisting of vertex-disjoint cycles of odd sizes (numbers of edges), then G is $\lceil \frac{2n}{3} \rceil$-approximately magic, for $n \geq 3$.*

Proof: Suppose that G is composed of l cycles C_1,C_2,...,C_l (of sizes $n_1, n_2, \ldots, n_l$, where $n_1 \geq n_2 \geq \ldots \geq n_l \geq 3$ are odd numbers, and $n_1+\cdots+n_l = n$). Let $n = 3t+\varepsilon$, $t \geq 1$, $\varepsilon \in \{0,1,2\}$. We partition the labels $1, \ldots, n$ into three groups $1, 2, \ldots, t$ and $t+1, \ldots, 2t+\varepsilon$ and $2t+\varepsilon+1, 2t+\varepsilon+2, \ldots, 3t+\varepsilon$. Let $A: a_1, a_2, \ldots, a_t$ denote the sequence $1, 2, \ldots, t$; let $B: b_1, b_2, \ldots, b_{t+\varepsilon}$ denote the sequence $2t+\varepsilon, 2t+\varepsilon-1, \ldots, t+1$; and let $C: c_1, c_2, \ldots, c_t$ denote the sequence $2t+\varepsilon+1, 2t+\varepsilon+2, \ldots, 3t+\varepsilon$. It is easy to see that $2t+\varepsilon+2 \leq a_i+c_j \leq 4t+\varepsilon$, $a_i+b_i = 2t+\varepsilon+1$, and $b_i+c_i = 4t+2\varepsilon+1$, for $i, j = 1, 2, \ldots, t$. In addition, $2t+3 \leq b_i+b_j \leq 4t+2\varepsilon-1$, for $i \neq j$, $i, j = 1, 2, \ldots, t$.

Let $m_i = \frac{n_i-1}{2}$, $i = 1, 2, \ldots, t$. We will present a labeling on G, which goes as follows. Label the cycles C_1,C_2,...,C_l one by one. For the i-th cycle C_i, pick the m_i smallest elements from the current (remained) A-sequence and the m_i smallest elements from the current (remained) C-sequence, if at this moment there are at least m_i elements remained in A (also C). Otherwise, pick all the remained elements of the two sequences. Specifically, we have the following two cases.

Case 1. At the beginning of the labeling of C_i, there are at least m_i elements remained in the current A (also C) sequence. Denote by $a_{s_i+1}, a_{s_i+2}, \ldots, a_{s_i+m_i}$ and $c_{s_i+1}, c_{s_i+2}, \ldots, c_{s_i+m_i}$ (where $s_1 = 0$, and $s_i = m_1+\cdots+m_{i-1}$ for $1 < i \leq l$) the m_i smallest elements of the current A (and C) sequence. Pick $b_{s_i+m_i}$ from the current B-sequence, and label the edges of C_i sequentially with $b_{s_i+m_i}$, c_{s_i+1}, a_{s_i+1}, c_{s_i+2}, a_{s_i+2},...,$c_{s_i+m_i}$, $a_{s_i+m_i}$, then remove these elements from their

sequences. Since $3t+\varepsilon+2 \le b_{s_i+m_i}+c_{s_i+1} \le 4t+2\varepsilon+1$, for the above labeling, each vertex sum of C_i is at least $2t+\varepsilon+1$, and at most $4t+2\varepsilon+1$.

Case 2. At the beginning of the labeling of C_i, the number of elements remained in the current A (also C) sequence is less than m_i. In this case we must have $n_1 \ge 5$ (otherwise all cycles are 'triangles', i.e. consisting of 3 edges, in our labeling each triangle will be labeled by three elements, and exactly one element from each sequence, which is a contradiction). Without loss of generality, we can assume that $l \ge 2$, since if $l=1$, G has been proved to be 2-approximately magic in Lemma 2.2.

If the current A (also C) sequence is empty, then label the remained non-labeled cycles arbitrarily using elements remained in B-sequence. Otherwise, pick all the elements $a_{s_i+1}, a_{s_i+2}, \ldots, a_t$ and $c_{s_i+1}, c_{s_i+2}, \ldots, c_t$ from the current A and C sequences. At this moment, besides b_t (where $t \ge 2$ since $l \ge 2$), b_1 is unused (if $i=1$, since $t \ge 2$, we have b_1 distinct from b_t and unused; if $i>1$, since $n_1 \ge 5$, b_1 has not been used for labeling C_1, thus is unused). Remove b_t and b_1 from the current B-sequence, and label the elements b_t, c_{s_i+1}, a_{s_i+1}, c_{s_i+2}, $a_{s_i+2}, \ldots, c_t$, a_t, b_1 sequentially to an arc of consecutive edges of C_i. Then, label the remained non-labeled edges of C_i using arbitrary elements remained in B-sequence, and remove these elements from B. Since $3t+\varepsilon+2 \le b_t+c_{s_i+1} \le 4t+2\varepsilon+1$, and $a_t+b_1 = 3t+\varepsilon$, we have that for the above labeling, each vertex sum of C_i is at least $2t+\varepsilon+1$, and at most $4t+2\varepsilon+1$.

Therefore, for the above labeling, the vertex sums of G are at least $2t+\varepsilon+1$ (which is $\lceil \frac{2n}{3} \rceil + 1$), at most $4t+2\varepsilon+1$ (which is $2\lceil \frac{2n}{3} \rceil + 1$), implying that the differences between vertex sums of G are at most $\lceil \frac{2n}{3} \rceil$. ■

Remark 5.2 $\lceil \frac{2n}{3} \rceil$ *obtained in Lemma 5.1 is actually asymptotically best possible. Consider the case that G is consisting of $\frac{n}{3}$ 'triangles'. Suppose that label 1 is assigned to an edge v_1v_2 of a triangle $v_1v_2v_3$, if the edge v_2v_3 or v_1v_3 is assigned with a label $l > \frac{2n}{3}$, then the difference of the two vertex sums of v_3 and v_1, or v_3 and v_2 will be at least $\frac{2n}{3}$. Similarly, suppose that label n is assigned to an edge v_4v_5 of a triangle $v_4v_5v_6$, if the edge v_4v_6 or v_5v_6 is assigned with a label $l \le \frac{n}{3}$, then the difference of the two vertex sums of v_5 and v_6, or v_4 and v_6 will be at least $\frac{2n}{3}$. If neither of the above two cases happens, then the vertex sum of v_1 or v_2 is at most $\frac{2n}{3}$, and the vertex sum of v_4 is at least $\frac{4n}{3}$, thus, the difference of the two vertex sums of v_4 and v_1, or v_4 and v_2 is at least $\frac{2n}{3}$.*

Now we will prove Theorem 1.4 for the case that k is even. Since k is even, G is an *even graph* (a graph with all vertices having even degrees), it follows that G can be decomposed into edge-disjoint simple cycles. In addition, two cycles having a common vertex can be merged into one circuit. Therefore, by repeating merging two cycles of odd sizes that having a common vertex into an even circuit, finally we will obtain a collection of s (≥ 0) even circuits P_1, P_2, $\ldots$, P_s (of sizes $2m_1, 2m_2, \ldots, 2m_s$), together with a collection of t (≥ 0) vertex-disjoint odd cycles Q_1, Q_2, $\ldots$, Q_t (of sizes $n_1, n_2, \ldots, n_t$, and $n_1+n_2+\cdots+n_t \le n$), such that each edge of G is in exactly one of these circuits or cycles.

Let $m = \frac{nk}{2}$ be the number of edges of G. First we label the even circuits P_1, P_2, $\ldots$, P_s. By viewing these circuits as cycles, using the 2-approximately magic labeling in Lemma 2.2, we assign labels $1, 2, \ldots, m_1$ and $m, m-1, \ldots, m-m_1+1$ to P_1, assign labels $m_1+1, m_1+2, \ldots, m_1+m_2$ and $m-m_1, m-m_1-1, \ldots, m-m_1-m_2+1$ to P_2, ..., and assign labels $m_1+\ldots+m_{s-1}+1$,

$m_1+\ldots+m_{s-1}+2$, ..., $m_1+\ldots+m_{s-1}+m_s$ and $m-m_1-\ldots-m_{s-1}$, $m-m_1-\ldots-m_{s-1}-1$, ..., $m-m_1-\ldots-m_{s-1}-m_s+1$ to P_s. Thus, the sum of any two consecutive edges of circuit P_i ($i=1,\ldots,s$) is among m, $m+1$, and $m+2$.

Let $m^*=m_1+m_2+\ldots+m_s$, and $n^*=n_1+n_2+\ldots+n_t$. If $n^*=0$ (i.e., there is no odd cycle), similarly as in Theorem 1.1, the above labeling of G can be proved to be k-approximately magic, by partitioning the k edges incident with any vertex of G into $k/2$ pairs such that each pair is composed of two consecutive edges in some circuit P_i ($i\in\{1,\ldots,s\}$). Otherwise, we have $n^*\geq 3$. Assign the remained labels $m^*+1, m^*+2,\ldots,m^*+n^*$ to the vertex-disjoint odd cycles Q_1, Q_2, ..., Q_t, using the $\lceil\frac{2n^*}{3}\rceil$-approximately magic labeling in Lemma 5.1. Since $2m^*+n^*=m$, and $\lfloor\frac{n^*}{3}\rfloor+\lceil\frac{2n^*}{3}\rceil=n^*$ for all integers $n^*\geq 1$, it follows that the sum of any two consecutive edges of these odd cycles is at least $2m^*+\lceil\frac{2n^*}{3}\rceil+1=m+1-\lfloor\frac{n^*}{3}\rfloor$ ($\leq m$), and at most $2m^*+2\lceil\frac{2n^*}{3}\rceil+1=m+1-\lfloor\frac{n^*}{3}\rfloor+\lceil\frac{2n^*}{3}\rceil$ ($\geq m+2$). Therefore, for the whole labeling of G, the sum received by any vertex of G is at least $m\times\frac{k-2}{2}+(m+1-\lfloor\frac{n^*}{3}\rfloor)$, at most $(m+2)\times\frac{k-2}{2}+(m+1-\lfloor\frac{n^*}{3}\rfloor+\lceil\frac{2n^*}{3}\rceil)$. Since $n^*\leq n$, the whole labeling of G is $(\frac{2n}{3}+k-1)$-approximately magic.

5.2 Proof of Theorem 1.5

If $k_1=2$, since $\frac{k_1^2}{2}>k_2$, $k_2=1$, G_2 is 1-regular, by Theorem 1.3, $G_1\times G_2$ is antimagic. In what follows we assume that $k_1\geq 3$.

We do the same labeling on $G_1\times G_2$ as in Theorem 1.2 (when k_1 is even, the labeling L_1 on G_1 here is $(\frac{2n_1}{3}+k_1-1)$-approximately magic). We will prove that for this labeling, (1) still holds if $k_1\geq 3$ is odd and $\frac{k_1^2-k_1}{2}\geq k_2$, or, if $k_1\geq 4$ is even and $\frac{k_1^2}{2}>k_2$.

$w(u_1,v_i)<w(u_2,v_i)<\ldots<w(u_{n_1},v_i)$ can be proved by using the same argument in Theorem 1.2, for $i=1,\ldots,n_2$. In order to prove $w(u_1,v_{i+1})-w(u_{n_1},v_i)>0$, for $i=1,\ldots,n_2-1$, there are two cases.

Case 1. k_1 is odd. Since G_1 is still $(\frac{n_1k_1}{2}-1)$-approximately magic, by using the same argument in Theorem 1.2, we can obtain that $w(u_1,v_{i+1})-w(u_{n_1},v_i)>0$, for $i=1,\ldots,n_2-1$.

Case 2. k_1 is even (thus $k_1\geq 4$). G_1 is $(\frac{2n_1}{3}+k_1-1)$-approximately magic. For each $i\in\{1,\ldots,n_2-1\}$, we have $w(u_1,v_{i+1})\geq w(u_1,v_i)+\frac{n_1k_1^2}{2}$ since $w_1(u_1,v_{i+1})=w_1(u_1,v_i)+\frac{n_1k_1^2}{2}$ and $w_2(u_1,v_{i+1})\geq w_2(u_1,v_i)$. In addition, $w(u_{n_1},v_i)\leq w(u_1,v_i)+(\frac{2n_1}{3}+k_1-1)+k_2(n_1-1)$ since $w_1(u_{n_1},v_i)\leq w_1(u_1,v_i)+(\frac{2n_1}{3}+k_1-1)$ and $w_2(u_{n_1},v_i)=w_2(u_1,v_i)+d(v_i)(n_1-1)\leq w_2(u_1,v_i)+k_2(n_1-1)$. Therefore, $w(u_1,v_{i+1})-w(u_{n_1},v_i)\geq(w(u_1,v_i)+\frac{n_1k_1^2}{2})-(w(u_1,v_i)+(\frac{2n_1}{3}+k_1-1)+k_2(n_1-1))=n_1(\frac{k_1^2}{2}-\frac{2}{3}-k_2)+k_2-k_1+1$.

Since $k_2<\frac{k_1^2}{2}$, there are two cases: $k_2\leq\frac{k_1^2}{2}-2$ or $k_2=\frac{k_1^2}{2}-1$. If $k_2\leq\frac{k_1^2}{2}-2$, $w(u_1,v_{i+1})-w(u_{n_1},v_i)\geq n_1(\frac{k_1^2}{2}-\frac{2}{3}-k_2)+k_2-k_1+1>n_1+k_2-k_1>0$ (since $n_1>k_1$). If $k_2=\frac{k_1^2}{2}-1$, $w(u_1,v_{i+1})-w(u_{n_1},v_i)\geq n_1(\frac{k_1^2}{2}-\frac{2}{3}-k_2)+k_2-k_1+1>\frac{k_1^2}{2}-k_1>0$ (since $k_1\geq 4$). Thus, in either case, we have $w(u_1,v_{i+1})-w(u_{n_1},v_i)>0$, for $i=1,\ldots,n_2-1$.

Therefore, (1) holds, the labeling for $k_1\geq 3$ is antimagic.

6 Concluding Remarks and Open Problems

Since the Eulerian circuit of an Eulerian graph (consequently the trails in the Listing Theorem) can be efficiently computed, the proofs in this paper provide efficient algorithms for finding the antimagic labelings.

It is easy to see that, for cycles, the 2-approximately magicness result in Lemma 2.2 is best possible (i.e., 2 can not be improved to 0 or 1). For n-vertex k-regular ($k > 2$) connected graphs, it may be interesting to prove that they are δ-approximately magic, where $\delta < (\frac{nk}{2} - 1)$ in case k is odd, or $\delta < k$ in case k is even, or, to prove some lower bounds on δ.

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